Coin tosses

(*) If a coin is tossed once, then $P(H) = P(T) = 50\%$.

Intuition: *If a coin is tossed many times, then about half the tosses will result in H and half will result in T.*

Key Question: How do we interpret the phrase '*about half the tosses*?'

There are two interpretations that make sense...

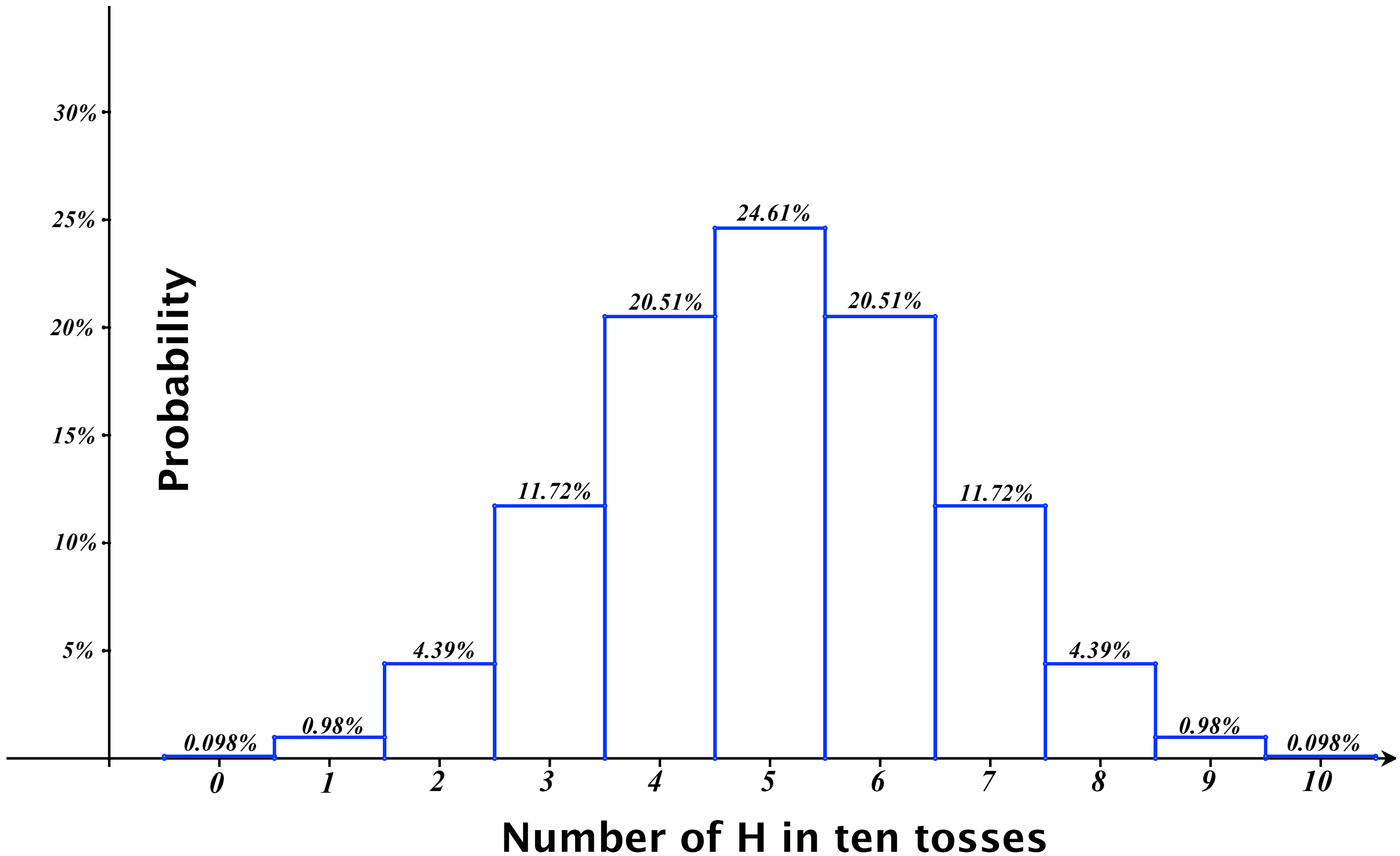
1. *If a coin is tossed n times, then about $n/2$ of the tosses result in H (and about $n/2$ result in T).*

2. *If a coin is tossed n times, then about 50% of the tosses result in H (and about 50% result in T).*

(*) These two interpretations are ***not*** exactly the same.

(*) The *law of averages* says that one of them is (almost) correct and one of them is not correct (not even almost).

If a fair coin is tossed 10 times, what will we see?



Probability histogram for the number of H in 10 tosses of a fair coin.

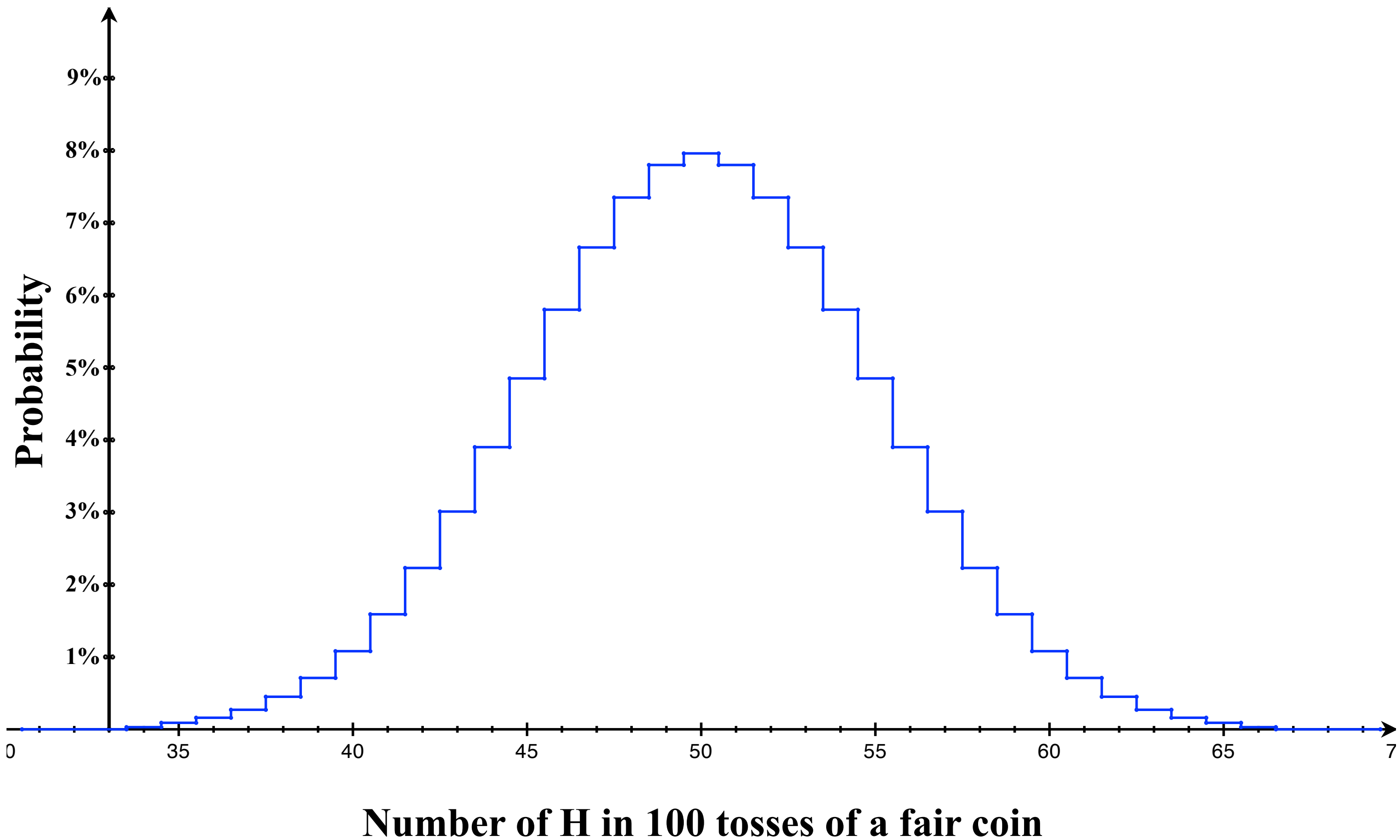
Observations:

- The probability of seeing exactly 5 H in 10 tosses is just below 25%.
- The probability that the number of H is between 4 and 6 is close to 66%.
- The probability that the number of H is between 3 and 7 is about 89%.
- We could summarize these numbers by saying that we will ***probably*** see ***about*** 5 H in 10 tosses of a fair coin.

$\Rightarrow$ We need to add the word '*probably*' to the sentence to give it a chance of being true in general.

More coin tosses

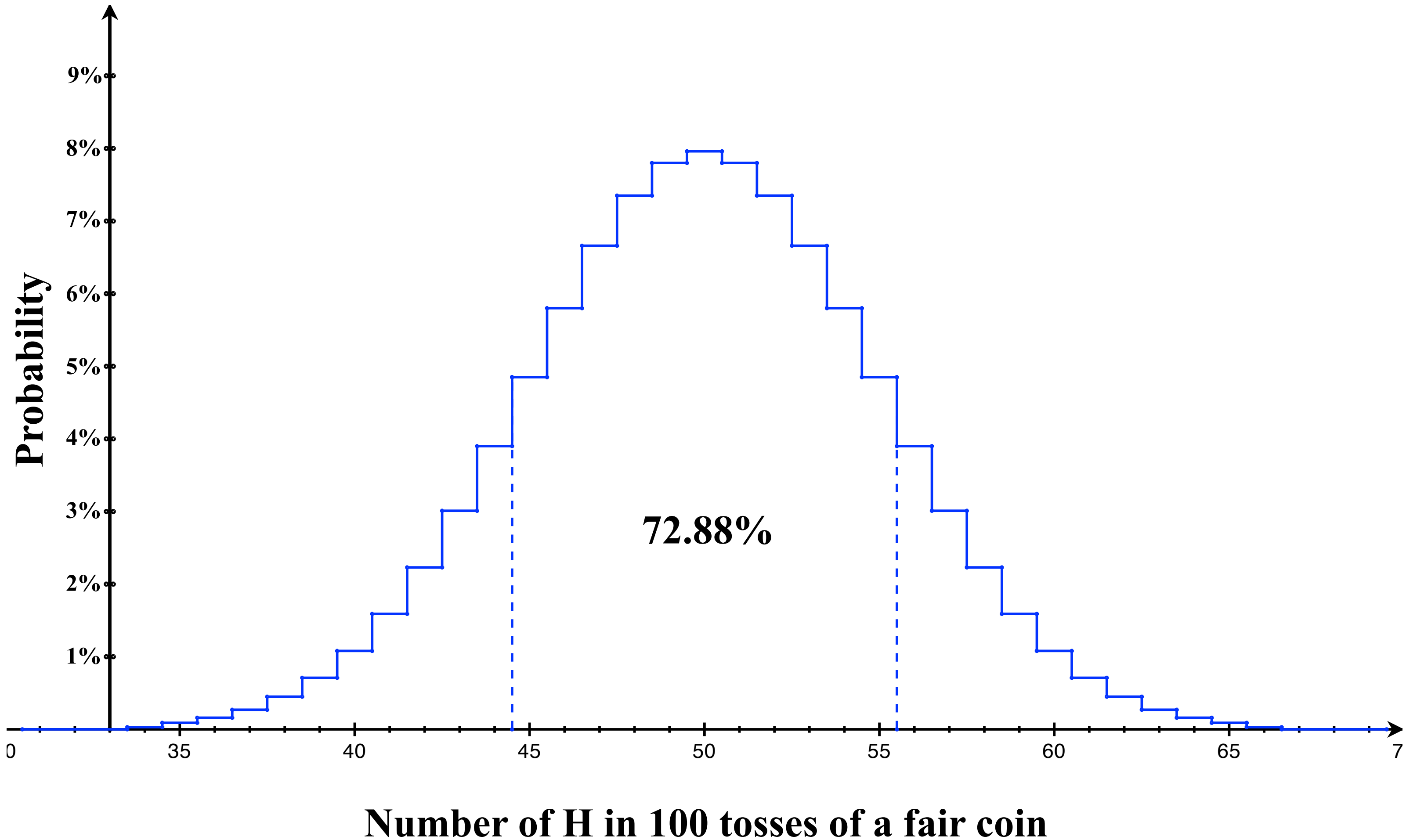
If a fair coin is tossed 100 times, what will we see?



Probability histogram for the number of H in 100 tosses of a fair coin.

Observations:

- The probability of seeing exactly 50 H in 100 tosses is 7.96%.
- The probability that the number of H in 100 tosses is between 49 and 51 is 23.56%.
- The probability that the number of H in 100 tosses is between 48 and 52 is 38.26%.
- The probability that the number of H in 100 tosses is between 47 and 53 is 51.58%.
- The probability that the number of H in 100 tosses is between 46 and 54 is 63.18%.



The probability that the number of H in 100 tosses is between 45 and 55 is 72.88%.

Analogously to the 10-toss scenario, we can say that in 100 tosses of a fair coin, we will ***probably*** see ***about*** 50 H.

Question: *In which of the two scenarios — 10 tosses or 100 tosses — is our prediction ('probably about half the tosses come up H) more accurate?*

Answer: *It depends on how we are measuring the accuracy.*

- In terms of the ***number*** of H, the prediction for 10 tosses gives a narrower range of possible values with a higher probability.
- In terms of the ***proportion*** of H, the prediction for 100 tosses gives a narrower range of possible ***percentages*** with higher probability.
  - The probability is 66% that ***percentage*** of H in 10 tosses will be between 40% and 60%.
  - The probability is 63.18% that the ***percentage*** of H in 100 tosses will be between 46% and 54%.
  - The probability is 72% that ***percentage*** of H in 100 tosses will be between 45% and 55%.

(*) When we toss a coin 10 times, the *expected number* of H is 5.

(*) When we toss a coin 100 times, the *expected number* of H is 50.

Question.

Why are the observed numbers of H (usually) different from the expected numbers?

Answer: *Chance error.*

Definition: The chance error is the difference between the observed number of H and the expected number of H.

Example: If we toss a coin 200 times and observe H 92 times, then

$$\begin{aligned}\text{Chance error} &= (\text{Observed \# of } H) - (\text{Expected \# of } H) \\ &= 92 - 100 = -8.\end{aligned}$$

$\Rightarrow$ To understand the *Law of Averages*, we need to understand the chance error.

(*) The expected number of H in n tosses is $\frac{n}{2}$.

(*) The *average size* of the chance error in n tosses is $\frac{\sqrt{n}}{2}$.

$\Rightarrow$ This *increases* with the number of tosses.

(*) The average *relative size* of the chance error in n tosses is

$$\frac{\sqrt{n}/2}{n} = \frac{1}{2\sqrt{n}}.$$

$\Rightarrow$ This *decreases* with the number of tosses.

Conclusion: As the number of tosses grows larger...

... the chance that the observed number of H is close to the expected number of H gets ***smaller***.

... the chance approaches 100% that the observed percentage of H is close to 50%. This is the *Law of averages*.

These phenomena are not limited to coin tosses...

Ten tickets are drawn at random with replacement from a box that contains three red tickets and seven blue tickets... (The '3R7B box')

Questions:

1. How many red tickets do you ***expect*** to see when you draw 10 tickets from the 3R7B box?

2. Why do we expect this number?

3. How ***accurate*** is the answer to the first question ***likely*** to be?

I.e., how close is the *observed* number of red tickets *likely* to be to the *expected* number of red tickets?

Answers:

1. We expect (about) 3 red tickets in 10 draws...

2. ...because it seems reasonable that the distribution of the observed tickets (red or blue) will match the distribution of tickets in the box.

3. To answer question 3., we need to find the probabilities of all of the possible outcomes.

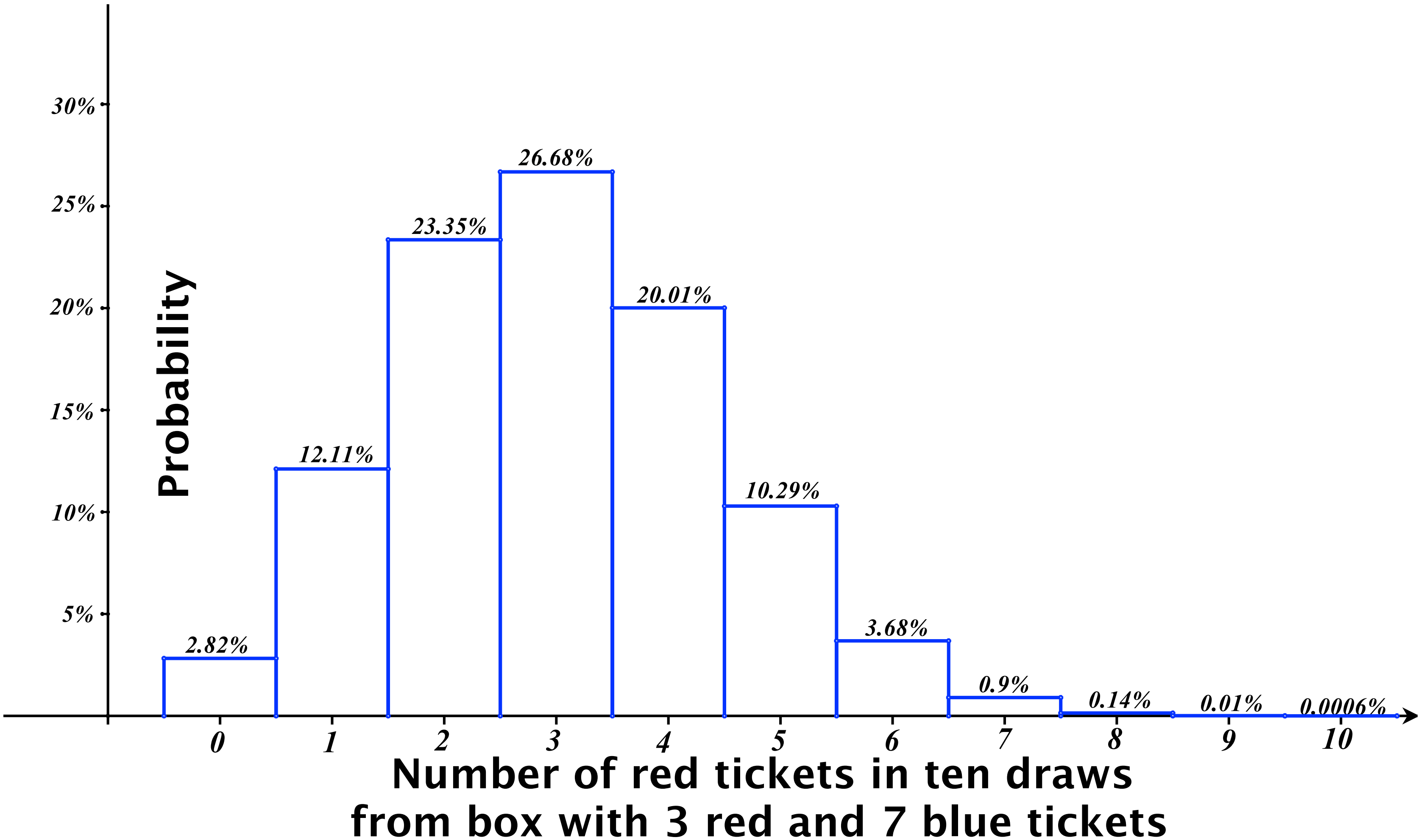
(*) The possible numbers of red tickets are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$, and for $0 \leq k \leq 10$,

$$P(k \text{ red tickets in 10 draws}) = \binom{10}{k}(0.3)^k(0.7)^{10-k}$$

E.g.,

$$P(2 \text{ red tickets in 10 draws}) = \binom{10}{2}(0.3)^2(0.7)^8 \approx 0.2335$$

We can collect all the probabilities in a histogram...



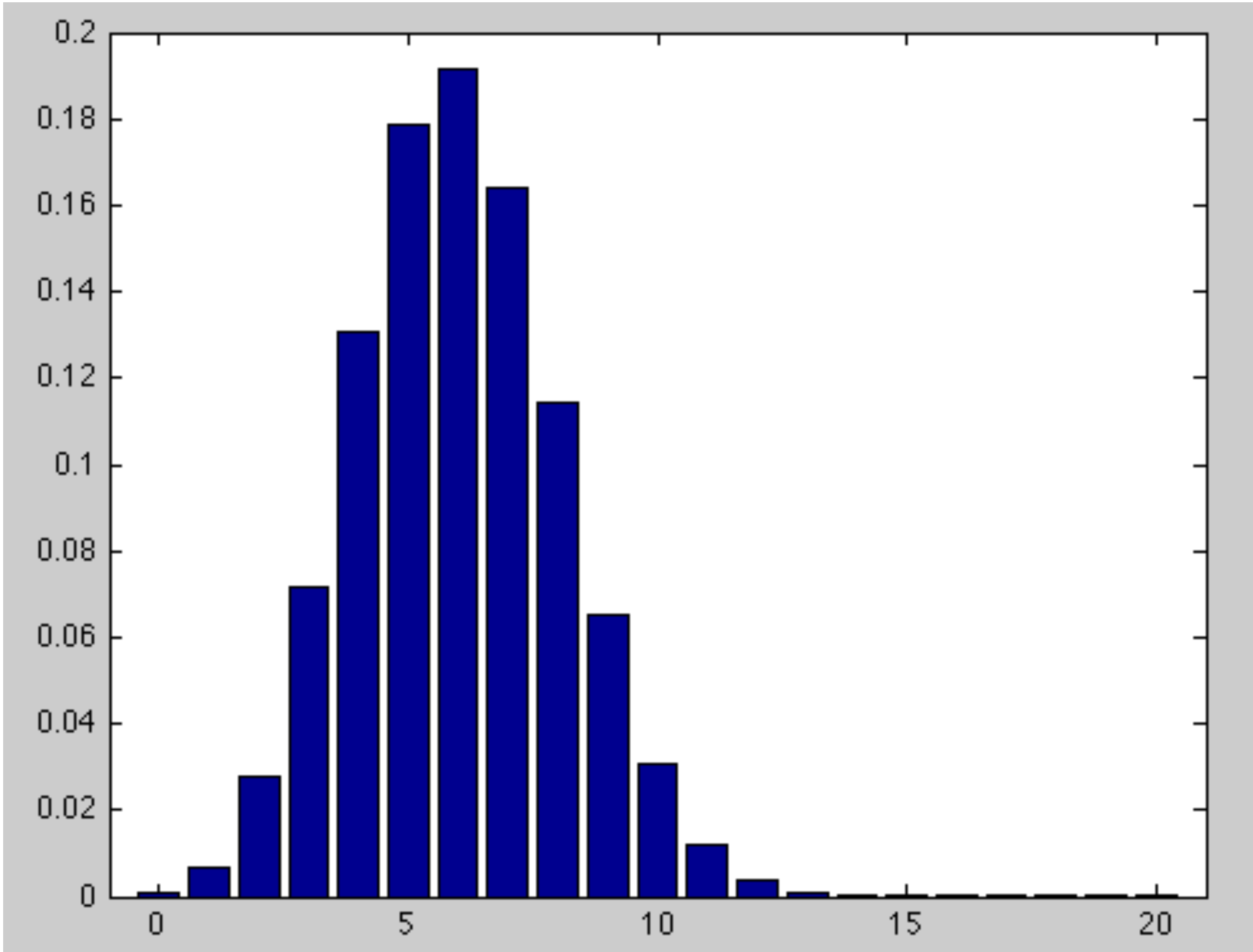
(*) The probability that the number of red tickets in 10 draws is between 2 and 4 ('about 3') is almost 70%.

(*) The small number of draws is a little misleading (as it was in the case of coin tosses), because...

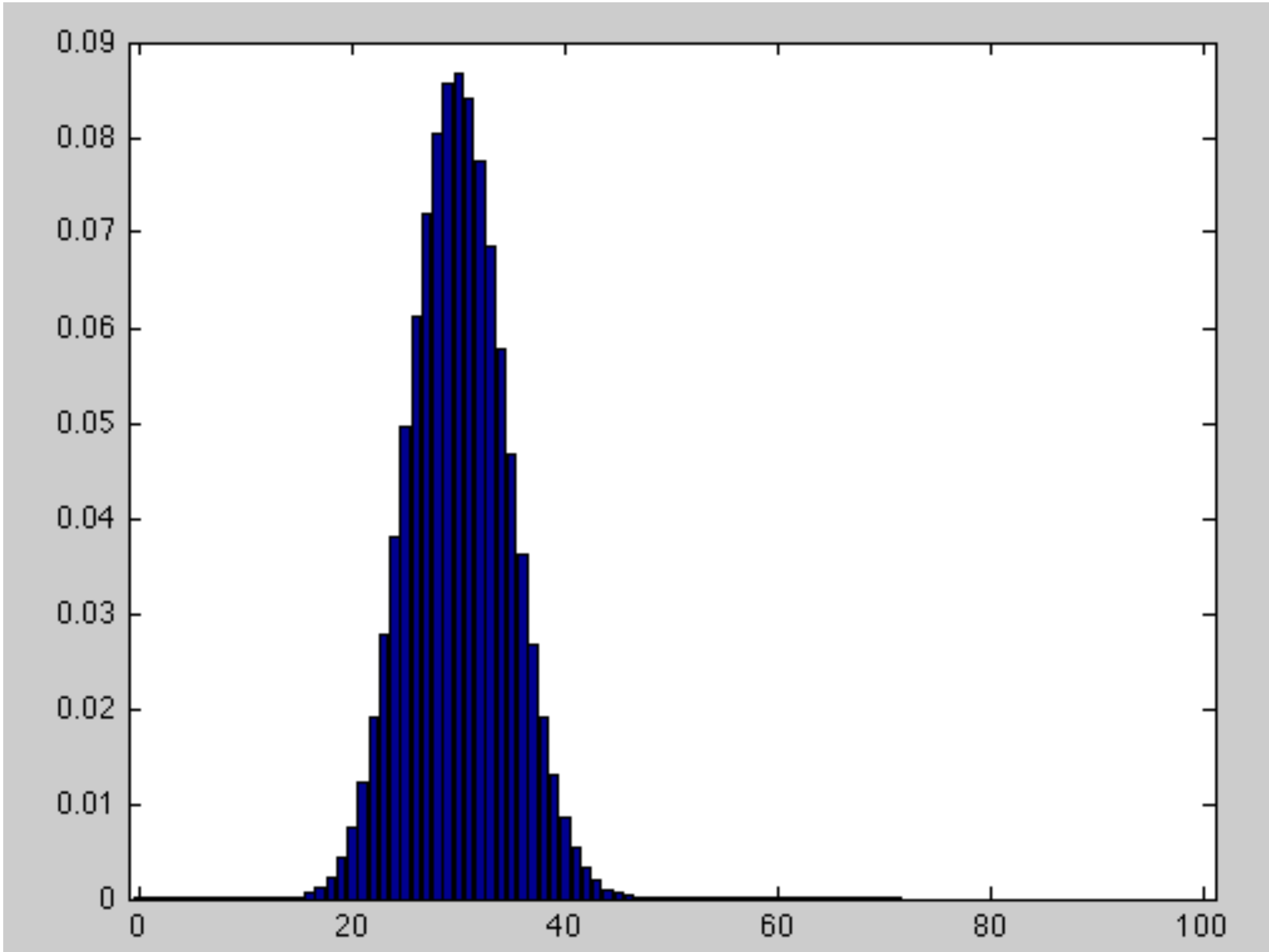
(*) ... *as the number of draws increases, the* ***chance error*** *also increases (on average).*

$$\text{chance error} = \text{observed number} - \text{expected number}.$$

(*) So, as the number of draws gets bigger, the observed number of red tickets is more and more likely to be farther and farther from the expected number of reds tickets.



Probability histogram for the number of red tickets observed in 20 random draws (with replacement) from the 3R7B box.



Probability histogram for the number of red tickets observed in 100 random draws (with replacement) from the 3R7B box.

Observations:

- The most likely number of red tickets in all three examples is

$$P(\text{red ticket in one draw}) \cdot (\text{number of draws}).$$

  This is the ***expected number*** of red tickets in each case.

- The probability that we see precisely the expected number of red tickets ***decreases*** as the number of draws increases. From about 27% in 10 draws, to about 19% in 20 draws, to about 8.5% in 100 draws, to about 2.75% in 1000 draws.

- The probability that the ***number*** of red tickets is close (e.g., within 2) of the expected number also decreases:

  $P(\text{between 1 and 5 red tickets in 10 draws}) = 92.45\%$

  $P(\text{between 4 and 8 red tickets in 20 draws}) = 77.96\%$

  $P(\text{between 28 and 32 red tickets in 100 draws}) = 41.43\%$

  $P(\text{between 298 and 302 red tickets in 1000 draws}) = 13.69\%$

As the number of draws from the box increases, the chance increases that the observed number of red tickets will deviate significantly from the expected number of red tickets:

- In 10,000 draws from the 3R7B box, the probability that the number of red tickets is ***more than*** 30 away from 3000 is about 50.57%
- In 1,000,000 draws from the 3R7B box, the probability that the number of red tickets is ***more than*** 300 away from 300,000 is about 51.2%.

On the other hand, in 10000 draws from the 3R7B box, the probability that the ***percentage*** of red tickets is in the range $30\% \pm 0.5\%$ is over 68%.

And even more strikingly, in 1000000 draws from the 3R7B box, the probability that the ***percentage*** of red tickets is in the range $30\% \pm 0.1\%$ is over 95%.

The Law of Averages (for the 3R7B box):

> *If many tickets are drawn at random with replacement from the 3R7B box, then it is very likely that about 30% of the tickets will be red.*

The "law of averages" does not say that...

- ...we will ***definitely*** see ***exactly*** 30% red tickets

  or

- ... we will ***probably*** see ***exactly*** 30% red tickets.

  or

- ... we will ***definitely*** see ***about*** 30% red tickets.

There is always a chance that the observed percentage of red tickets will be far from the expected percentage, even for an extremely large number of draws.

The law of averages, more generally:

If tickets are drawn from a box containing $\boxed{1}$ *s and* $\boxed{0}$ *s, then as the number of draws increases, the probability approaches 100% that the observed percentage of* $\boxed{1}$ *s is very close to the percentage of* $\boxed{1}$ *s in the box.*

Comments:

- The law is true for draws with replacement and for draws without replacement. The results are even sharper when the draws are done without replacement (because the chance error is smaller in this case).

- The difference between the observed ***number*** of red tickets and the expected ***number*** of red tickets is *likely to get bigger* as the number of draws (with replacement) grows.

- The law of averages does ***not*** say anything about what will happen on the ***next draw***.