SD vs. SD^+

One of the most important uses of *sample statistics* is to ***estimate*** the corresponding *population parameters*.

- The mean of a ***representative*** sample is a good estimate of the mean of the population that the sample represents.
- The SD of a ***representative*** sample tends to ***underestimate*** the SD of the population from which it was drawn.
- To correct for this, statisticians use the SD^+ of the sample to estimate the SD of the population. If n is sample size, then

$$SD^+ = \sqrt{\frac{n}{n-1}} \cdot SD_{\text{sample}} = \sqrt{\frac{1}{n-1}\sum(x_j - \overline{x})^2}$$

- If the sample size is large, then there is no significant difference between SD and SD^+ because $\sqrt{n/(n-1)} \approx 1$ when n is large.
- The SD^+ is called the *sample standard deviation*.

Question: *How is the data clustered around the mean?*

The answer is easiest to give in terms of the standard deviation:

In ***any*** *data set, the proportion of the data that lies more than k SDs from the mean is always less than $1/k^2$.*

This fact is known as ***Chebyshev's inequality***, and follows directly from how the standard deviation is defined (with a little work and cleverness).

For example, less than $1/4 = 25\%$ of the values ***in any data set*** lie more than 2 SDs from the average value (mean). Less than $1/9 \approx 11.11\%$ of the data lie more than 3 SDs from the average value. Etc.

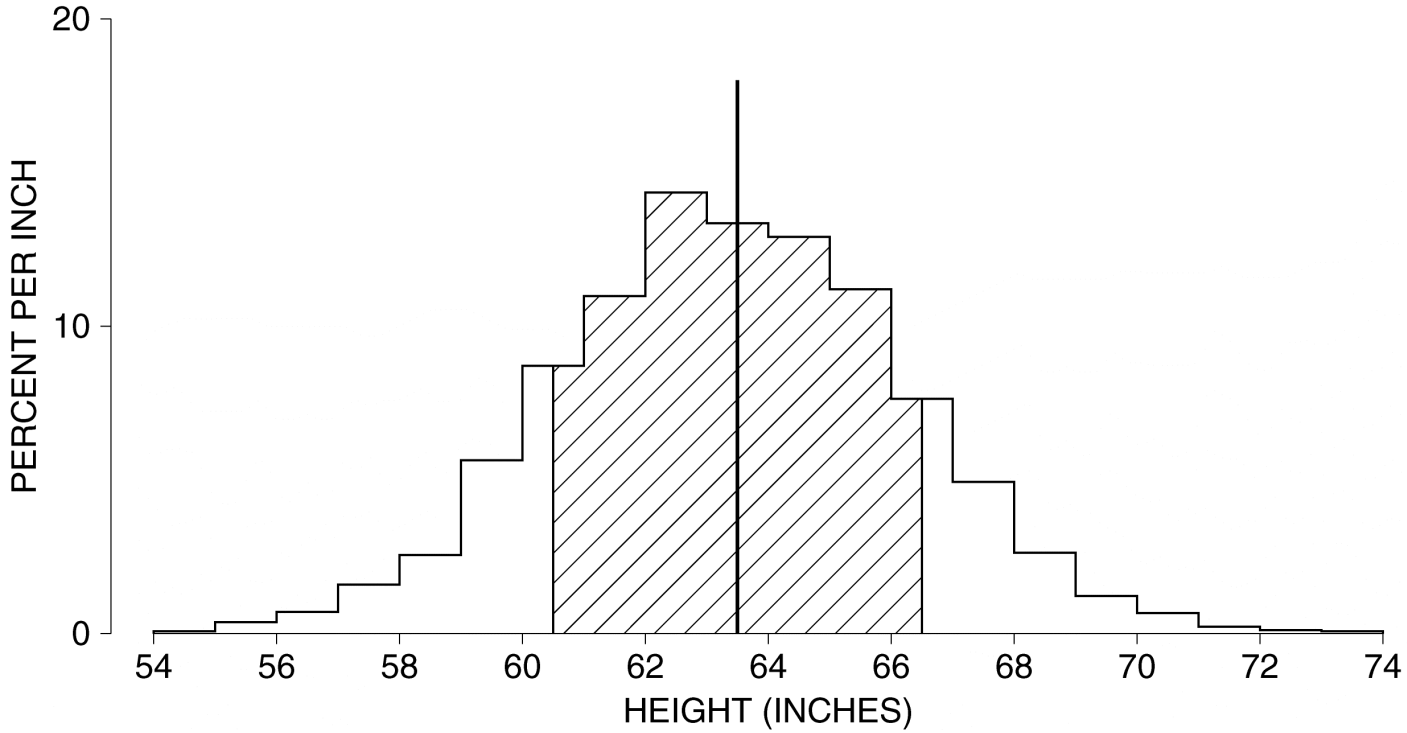
Or we can say that in any data set, more than 75% of the data lie within 2 SDs of the mean, and more than 88.88% of the data lie within 3 SDs of the mean.

The estimates above are true for ***any*** set of data. On the other hand, if we know more about the data, then we can often get sharper estimates.

For certain types of data sets, almost all of the data lies within two or three SDs of the average.

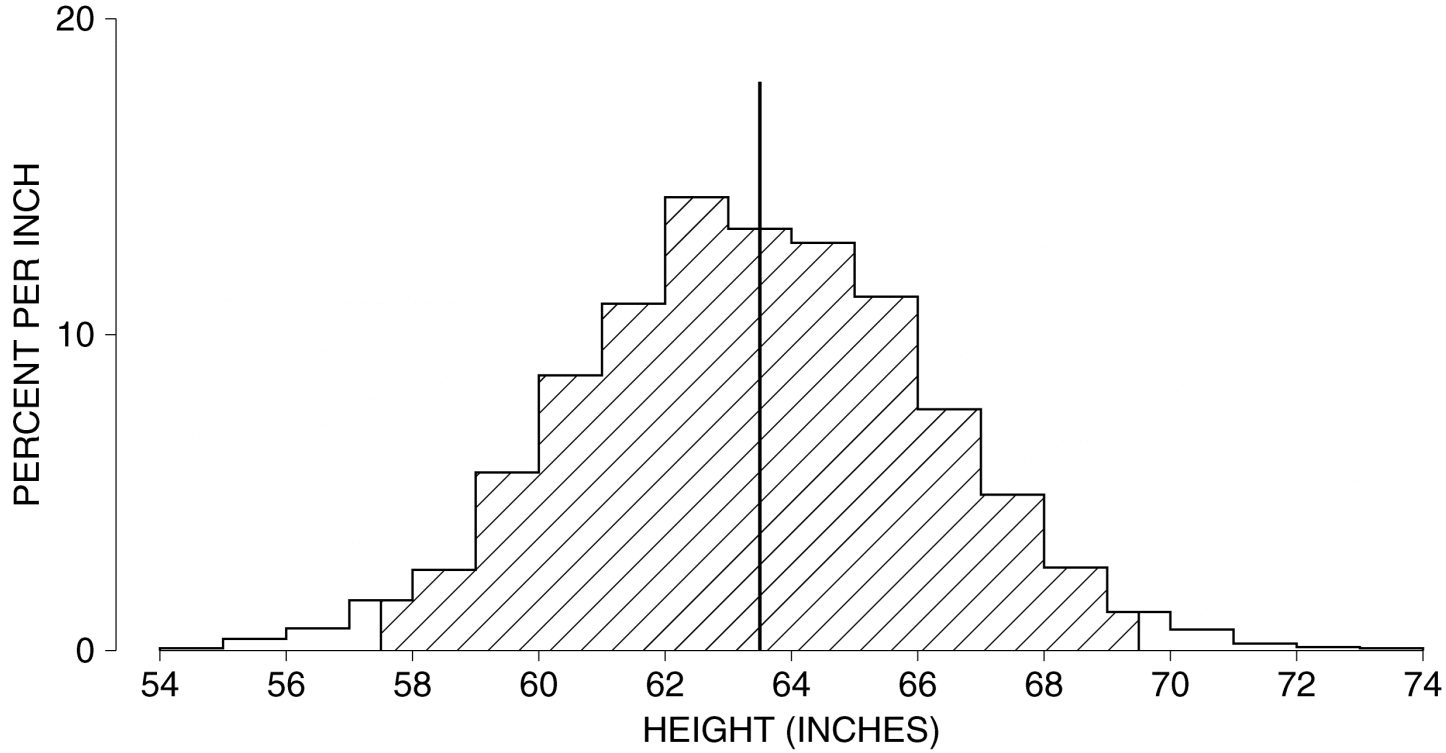
Example (from the book): $\overline{h} = 63.5$ inches and $SD_h \approx 3$ inches...

Figure 8. The SD and the histogram. Heights of 2,696 women age 18 and over in HANES5. The average of 63.5 inches is marked by a vertical line. The region within one SD of the average is shaded: 72% of the women differed from average by one SD (3 inches) or less.



Example (continued): $\overline{h} = 63.5$ inches and $SD_h \approx 3$ inches...

Figure 9. The SD and the histogram. Heights of 2,696 women age 18 and over in HANES5. The average of 63.5 inches is marked by a vertical line. The region within two SDs of the average is shaded: 97% of the women differed from average by two SDs (6 inches) or less.



To better understand clustering around the mean, we introduce

Standard units

If x_j comes from data with average $\overline{x}$ and standard deviation SD_x, we convert x_j to its standard unit, z_j, by setting

$$z_j = \frac{x_j - \overline{x}}{SD_x}.$$

(*) $|z_j|$ tells us how far x_j is from $\overline{x}$ as a multiple of SD_x.

(*) If $z_j > 0$, then x_j is above average; if $z_j < 0$, then x_j is below average.

(*) Standard units are 'pure numbers': there are no units of measurement (inches, dollars, etc.) associated with standard units.

$\Rightarrow$ The units of the x_js are cancelled when we divide $x_j - \overline{x}$ by SD_x.

(*) The standard unit value z_j of a given datum x_j is also called the ***z-score*** of x_j.

Example. Suppose that the average January temperature in Podunk is 45 °F , with an SD of 2 °F, while in Whoville the average January temperature is 25 °F with an SD of 5 °F. On January 20th, the temperature in Whoville was 16 °F and in Podunk it was 38 °F.

Where was the temperature more *unusual* that day?

We can answer this by converting the temperatures on January 20th in both towns to standard units:

$$z_p = \frac{38 - 45}{2} = -3.5 \quad \text{and} \quad z_w = \frac{16 - 25}{5} = -1.8.$$

(*) Both temperatures were below average.

(*) The *z-score* for Podunk is more negative than the *z-score* for Whoville, so from a statistical point of view the temperature in Podunk was more unusual that day.

Intuition: *The larger $|z_j|$, the more unusual x_j is (in the set of data it came from).*

Observation. *Converting* ***any*** *set of data,* $\{x_1, x_2, \ldots, x_n\}$ *with average* $\overline{x}$ *and standard deviation* $SD_x = s$*, to standard units produces a set of numbers* $\{z_1, z_2, \ldots, z_n\}$ *with average* $\overline{z} = 0$ *and standard deviation* $SD_z = 1$*.*

Because arithmetic...

$$\begin{aligned}
\overline{z} &= \frac{z_1 + z_2 + \cdots + z_n}{n} \\
&= \frac{\frac{x_1-\overline{x}}{s} + \frac{x_2-\overline{x}}{s} + \cdots + \frac{x_n-\overline{x}}{s}}{n} \\
&= \frac{\frac{x_1-\overline{x}}{n} + \frac{x_2-\overline{x}}{n} + \cdots + \frac{x_n-\overline{x}}{n}}{s} \\
&= \frac{\frac{x_1+x_2+\cdots+x_n}{n} - \frac{\overbrace{\overline{x} + \overline{x} \cdots + \overline{x}}^{n}}{n}}{s} \\
&= \frac{\overline{x} - \overline{x}}{s} = 0
\end{aligned}$$

and more arithmetic

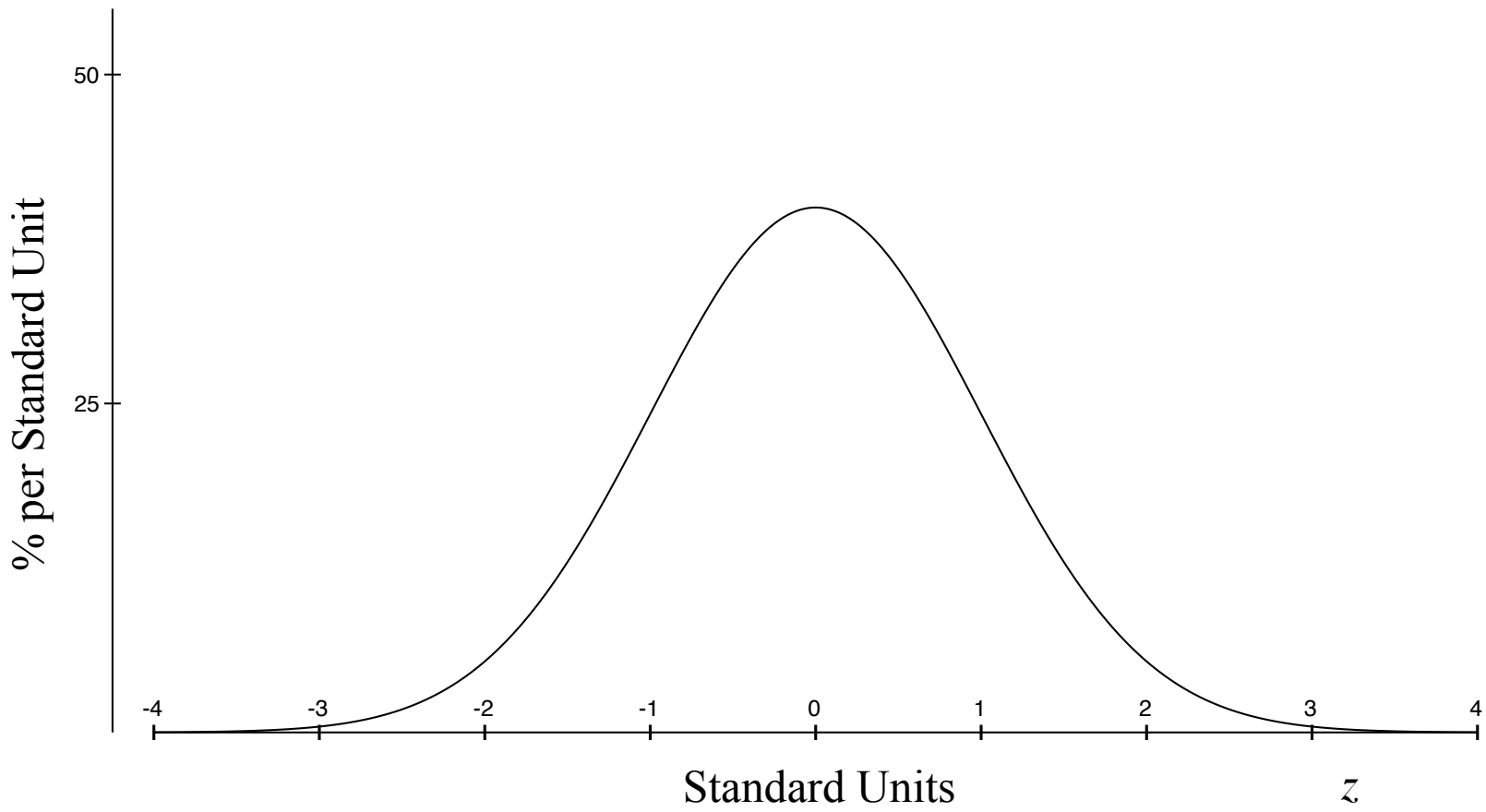
$$
\begin{aligned}
SD_z &= \sqrt{\frac{z_1^2 + z_2^2 + \cdots + z_n^2}{n}} \\
&= \sqrt{\frac{\left(\frac{x_1-\overline{x}}{s}\right)^2 + \left(\frac{x_2-\overline{x}}{s}\right)^2 + \cdots + \left(\frac{x_n-\overline{x}}{s}\right)^2}{n}} \\
&= \sqrt{\frac{\frac{(x_1-\overline{x})^2}{s^2} + \frac{(x_2-\overline{x})^2}{s^2} + \cdots + \frac{(x_n-\overline{x})^2}{s^2}}{n}} \\
&= \sqrt{\frac{\frac{(x_1-\overline{x})^2+(x_2-\overline{x})^2+\cdots+(x_n-\overline{x})^2}{n}}{s^2}} \\
&= \frac{\sqrt{\frac{(x_1-\overline{x})^2+(x_2-\overline{x})^2+\cdots+(x_n-\overline{x})^2}{n}}}{\sqrt{s^2}} \\
&= \frac{s}{s} = 1
\end{aligned}
$$

The normal approximation, I

- Different sets of data may be seen to have very similar distributions, *once they have been converted to standard units.*
- Converting to standard units moves the center of the histogram (the average of the data) to 0, and scales the data as a whole so that one SD is converted to 1 unit.
- In many cases, the histogram of the data, once converted to standard units, takes on a somewhat *bell-shaped* form — the form of the ***normal curve***.
- The normal curve is the graph of the function

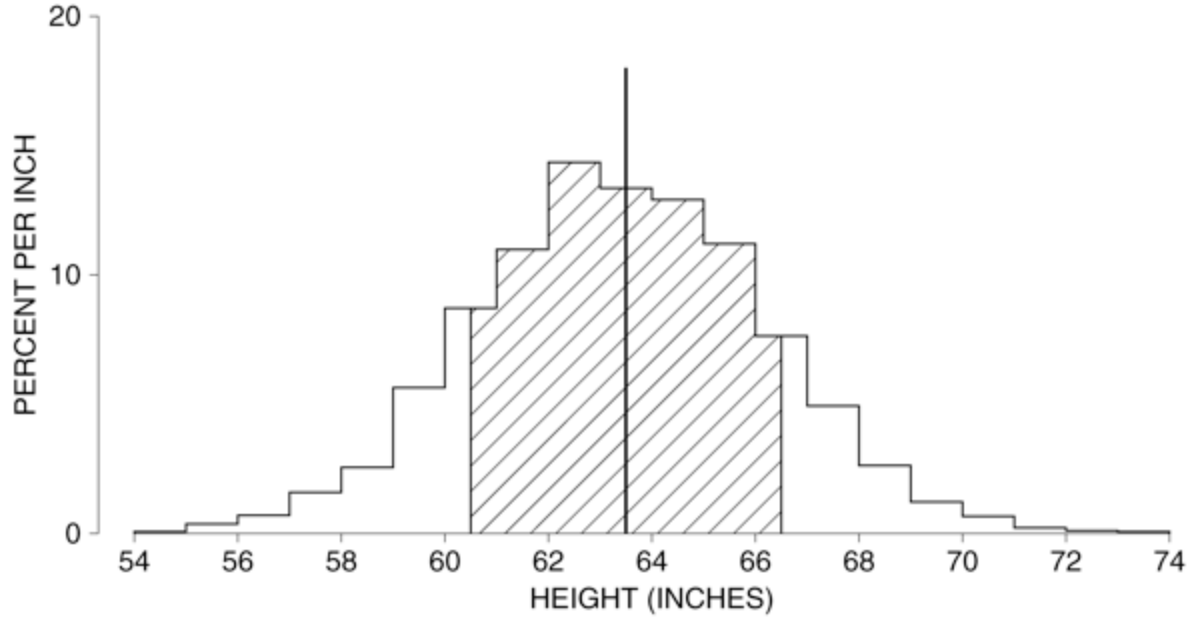
$$y = \frac{1}{\sqrt{2\pi}} e^{-z^2/2},$$

(where $e = 2.7182818\ldots$).



The normal curve is symmetric around the line $z = 0$, and the total area under the curve is equal to 1 (or 100%, if you prefer).

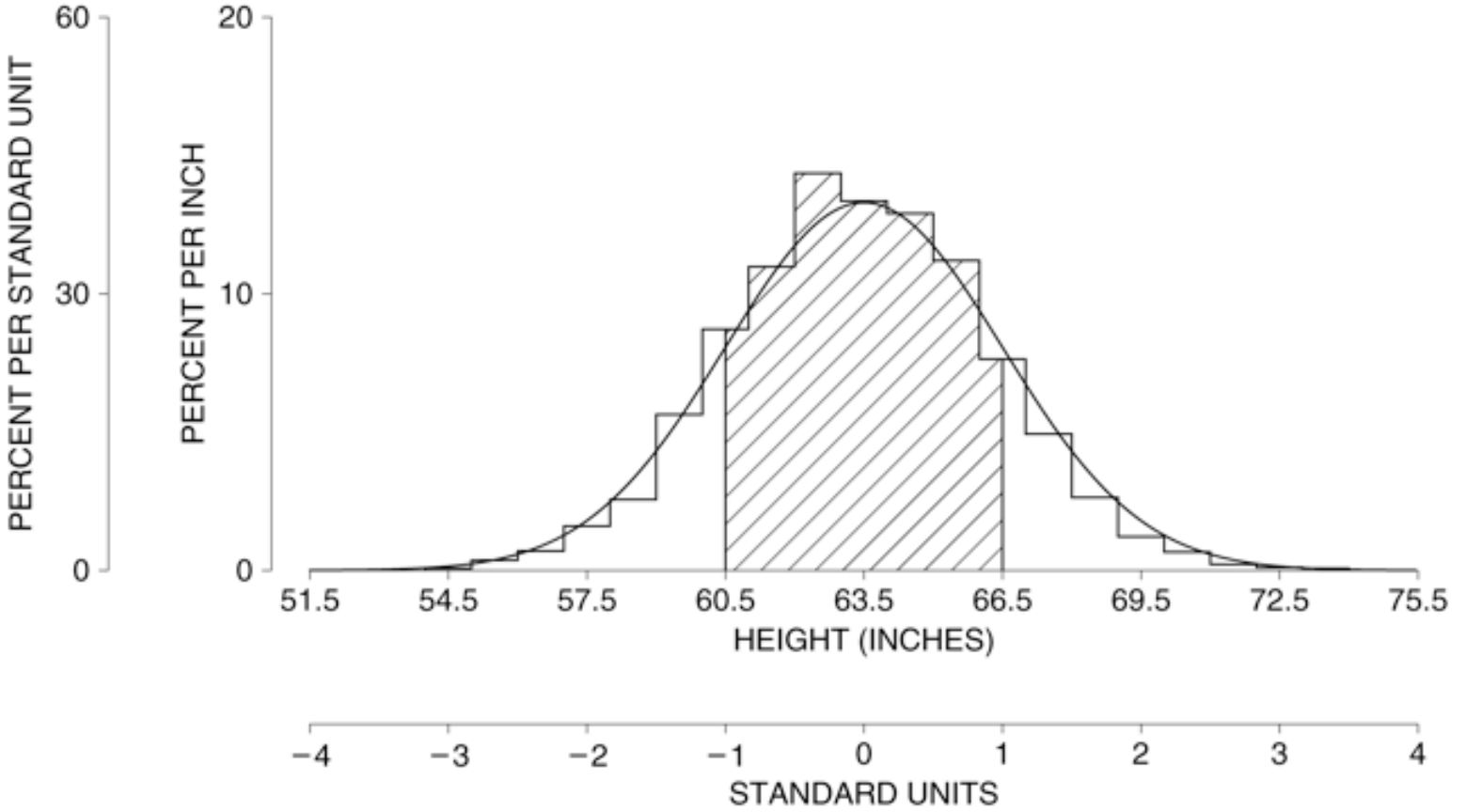
Example: The distribution of heights of women age 18 and over in HANES5 (Health and Nutrition Examination Study, '03 - '04) appears in the histogram below (from page 81 in chapter 5 of FPP).



The average height is 63.5 and the SD is about 3. The shaded region represents the heights that fall within one SD of average.

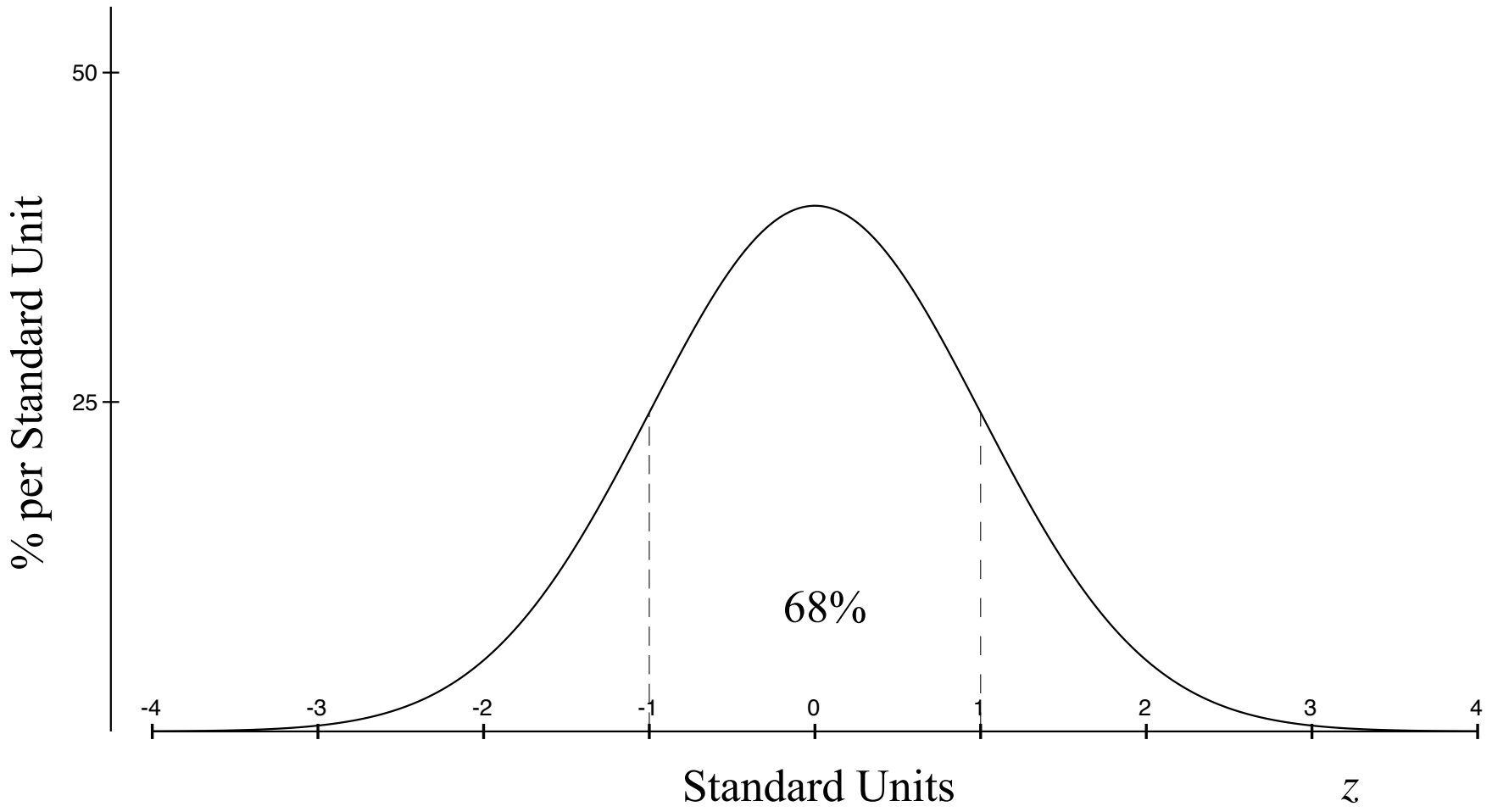
To see how well the distribution of the height data is approximated by the normal curve, we must convert the data to standard units and sketch the histogram for the 'standardized' (or normalized) data.

To save a lot of drawing time... notice that the conversion to standard units is just a *rescaling*. This means that instead of converting all of the heights to their standard units and then drawing a new histogram, we can instead ***change the horizontal and vertical scales on the original histogram***.

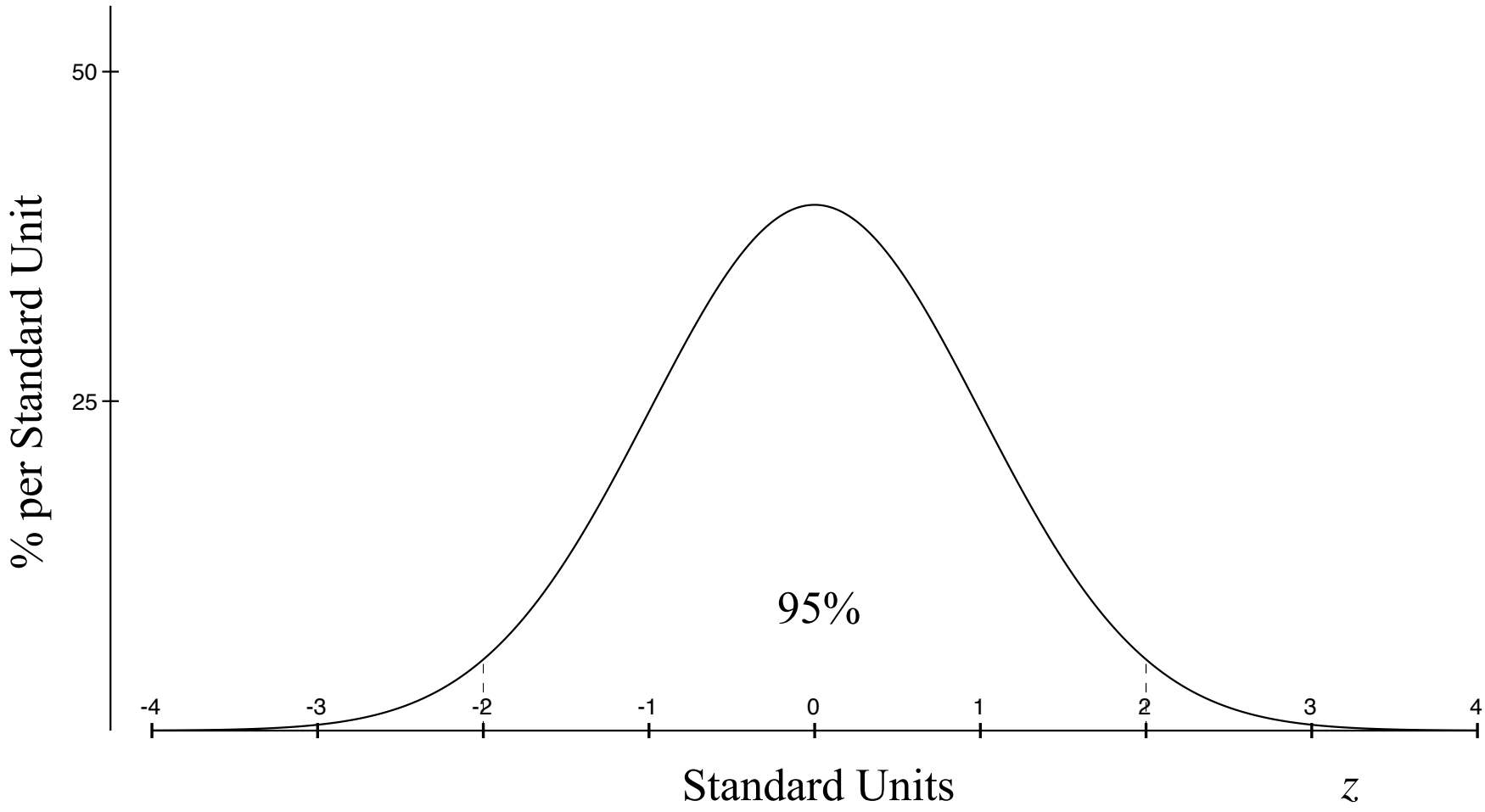


- If the (rescaled) histogram is well-approximated by the normal curve, then area of regions under the histogram will be approximately equal to areas under the normal curve for the same range of standard units.
- I.e., the percentage of the data that lies within 1 SD of the average will be approximately equal to the area under the normal curve between -1 and 1; the percentage of the data lying within 2 SDs of the average will be approximately equal to the area under the normal curve between -2 and 2; and so forth.
- This is useful, because the distribution of the area under the normal curve is well-understood.

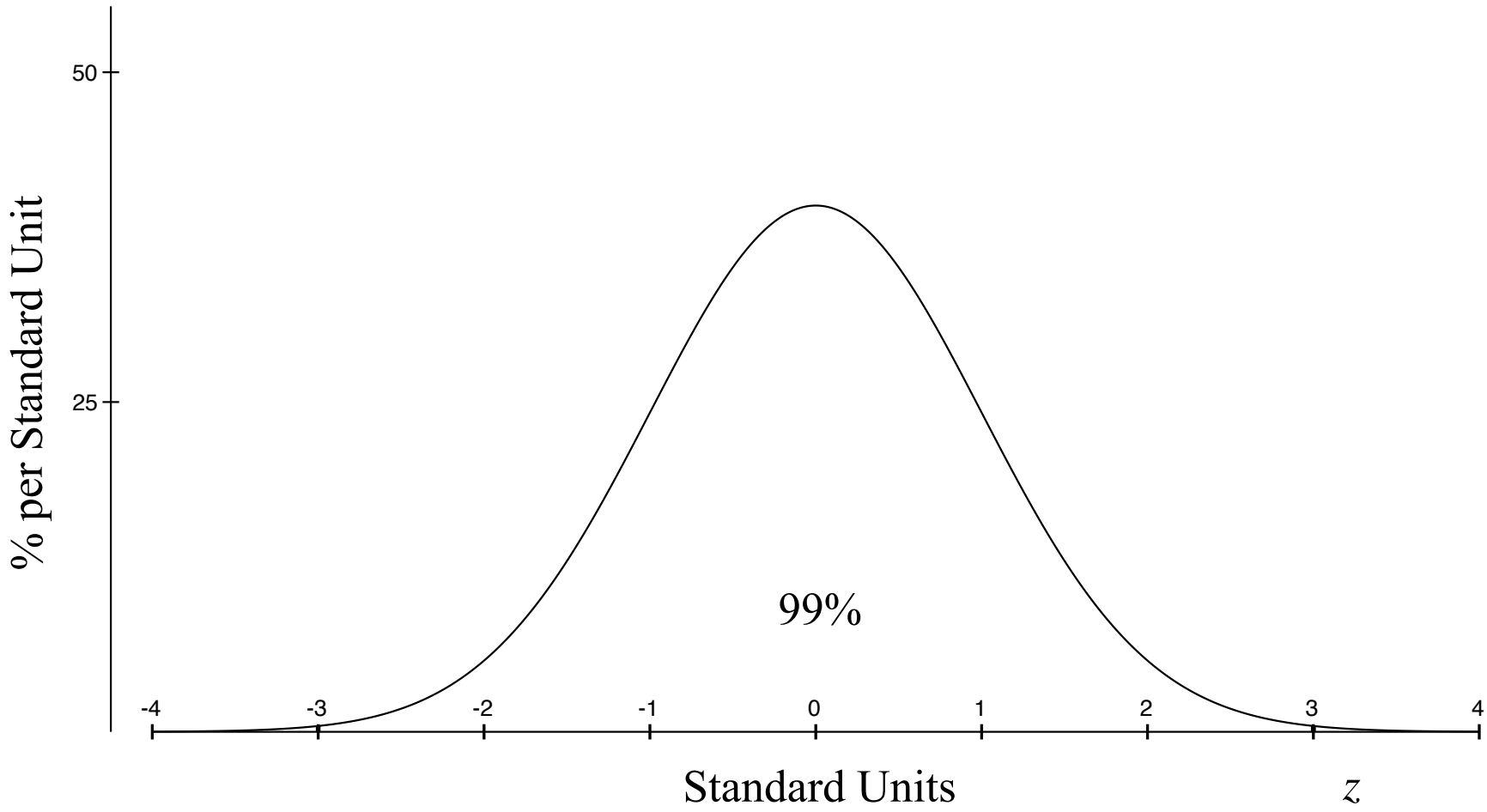
In particular...



(*) The area under the normal curve between -1 and 1 is $\approx 0.68 = 68\%$.



(*) The area under the normal curve between -2 and 2 is $\approx 0.95 = 95\%$.



(*) The area under the normal curve between -3 and 3 is $\approx 0.99 = 99\%$.

'Rule of thumb':

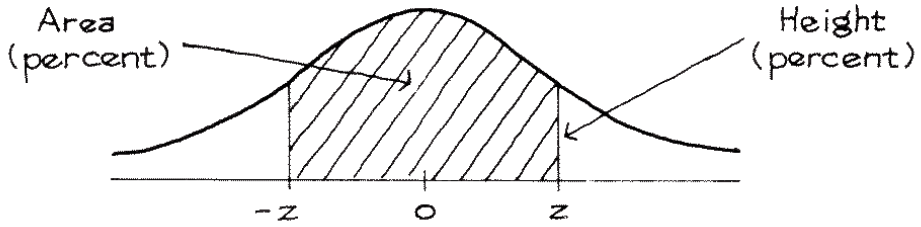
If a set of data has an approximately normal distribution, then:

- ***About*** 68% ***of the data lies within one SD of average;***
- ***About*** 95% ***of the data lies within two SDs of average;***
- ***About*** 99% ***of the data lies within three SDs of average;***

Remember: This rule ***only applies to data that is (approximately) normally distributed!*** Absent that condition (or assumptions about how the data is distributed) we rely on weaker (but more general) estimates (like Chebychev's inequality).

To calculate areas under the normal curve for regions other than those above (-1 to 1, -2 to 2 and -3 to 3), we use a ***normal table***, like the one found in the back of the textbook.

A normal table



A NORMAL TABLE

z	*Height*	*Area*	*z*	*Height*	*Area*	*z*	*Height*	*Area*
0.00	39.89	0	1.50	12.95	86.64	3.00	0.443	99.730
0.05	39.84	3.99	1.55	12.00	87.89	3.05	0.381	99.771
0.10	39.69	7.97	1.60	11.09	89.04	3.10	0.327	99.806
0.15	39.45	11.92	1.65	10.23	90.11	3.15	0.279	99.837
0.20	39.10	15.85	1.70	9.40	91.09	3.20	0.238	99.863
0.25	38.67	19.74	1.75	8.63	91.99	3.25	0.203	99.885
0.30	38.14	23.58	1.80	7.90	92.81	3.30	0.172	99.903
0.35	37.52	27.37	1.85	7.21	93.57	3.35	0.146	99.919
0.40	36.83	31.08	1.90	6.56	94.26	3.40	0.123	99.933
0.45	36.05	34.73	1.95	5.96	94.88	3.45	0.104	99.944

0.50	35.21	38.29	2.00	5.40	95.45	3.50	0.087	99.953
0.55	34.29	41.77	2.05	4.88	95.96	3.55	0.073	99.961
0.60	33.32	45.15	2.10	4.40	96.43	3.60	0.061	99.968
0.65	32.30	48.43	2.15	3.96	96.84	3.65	0.051	99.974
0.70	31.23	51.61	2.20	3.55	97.22	3.70	0.042	99.978
0.75	30.11	54.67	2.25	3.17	97.56	3.75	0.035	99.982
0.80	28.97	57.63	2.30	2.83	97.86	3.80	0.029	99.986
0.85	27.80	60.47	2.35	2.52	98.12	3.85	0.024	99.988
0.90	26.61	63.19	2.40	2.24	98.36	3.90	0.020	99.990
0.95	25.41	65.79	2.45	1.98	98.57	3.95	0.016	99.992
1.00	24.20	68.27	2.50	1.75	98.76	4.00	0.013	99.9937
1.05	22.99	70.63	2.55	1.54	98.92	4.05	0.011	99.9949
1.10	21.79	72.87	2.60	1.36	99.07	4.10	0.009	99.9959
1.15	20.59	74.99	2.65	1.19	99.20	4.15	0.007	99.9967
1.20	19.42	76.99	2.70	1.04	99.31	4.20	0.006	99.9973
1.25	18.26	78.87	2.75	0.91	99.40	4.25	0.005	99.9979
1.30	17.14	80.64	2.80	0.79	99.49	4.30	0.004	99.9983
1.35	16.04	82.30	2.85	9.69	99.56	4.35	0.003	99.9986
1.40	14.97	83.85	2.90	0.60	99.63	4.40	0.002	99.9989
1.45	13.94	85.29	2.95	0.51	99.68	4.45	0.002	99.9991

Using the normal table

(i) The table in the appendix gives the areas for symmetric regions $-z_0 \leq z \leq z_0$ (as *percentages*), where $0 \leq z_0 \leq 4.45$. If $z_0 \geq 4.50$, you can assume that the corresponding area is 99.9999%.

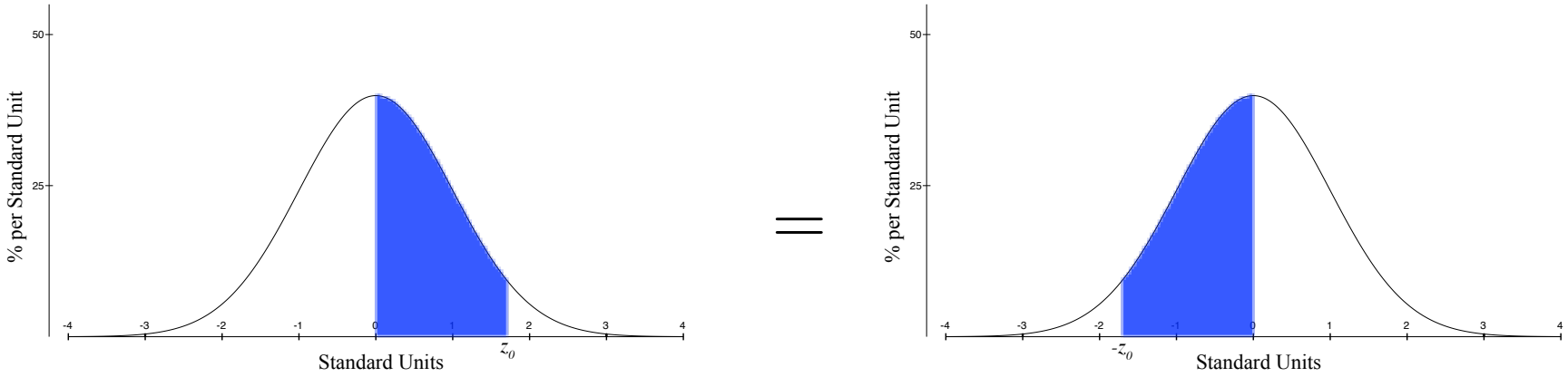
Example: Suppose that the heights of men aged 25 – 35 in a certain city are distributed (approximately) normally with an average of 67 inches and a standard deviation of 2.5 inches.

What percentage of these men are between 65 and 69 inches tall?

a. A height of 65 inches corresponds to $\frac{65-67}{2.5} = -0.8$ standard units, and 69 inches corresponds to $\frac{69-67}{2.5} = 0.8$ standard units.

b. The percentage we want is (approximately) equal to the area under the normal curve between -0.8 and 0.8 which is equal to the table entry for $z_0 = 0.8$, which is 57.63%.

(ii) The normal curve is ***symmetric around z = 0*** so the area under the curve between 0 and z_0 is equal to the area under the curve between $-z_0$ and 0, and both are equal to exactly one half the table entry for z_0.

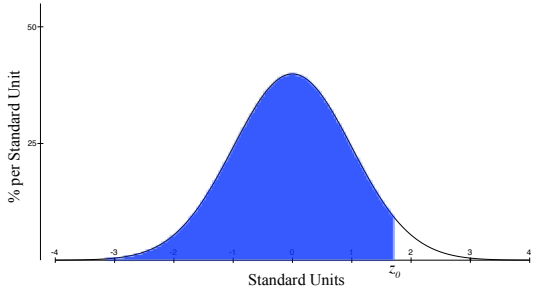


Example. What percentage of the men in the previous example are between 67 and 70 inches tall?

a. 67 inches is average which corresponds to 0 standard units and 70 inches corresponds to $\frac{70-67}{2.5} = 1.2$ standard units.

b. The percentage we want is (approximately) equal to the area under the normal curve between 0 and 1.2 which is equal to ***half*** the table entry for $z_0 = 1.2$. This is $76.99/2\% \approx 38.5\%$.

(iii) If $z_0 > 0$, then the area under the normal curve *to the left of* z_0 is equal to 50% plus half the table entry for z_0, because...



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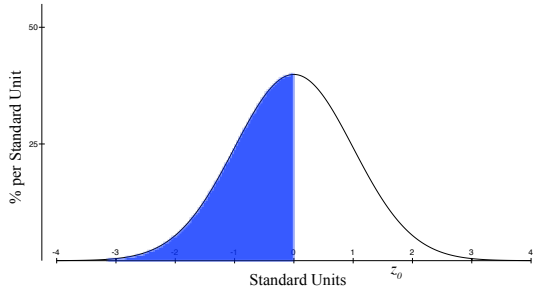

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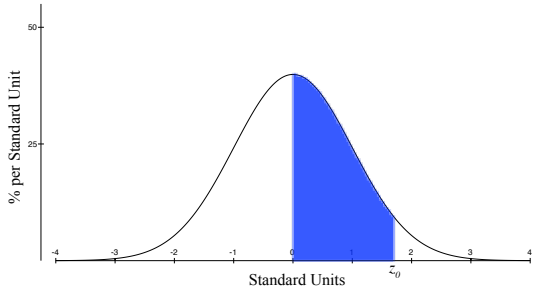
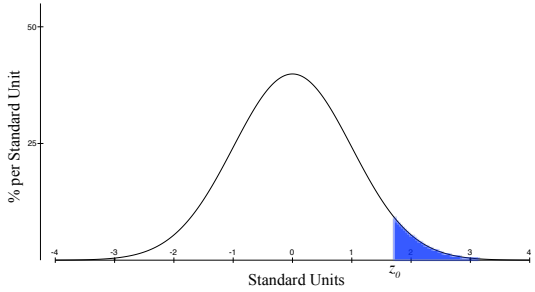


$= 50\% + \frac{1}{2}\text{Table}(z_0).$

Example. What percentage of the men in the previous examples are less than six feet, two inches tall?

Six feet, two inches is 74 inches which corresponds to $\frac{74-67}{2.5} = 2.8$ standard units. The table entry for 2.8 is 99.49%, so the percentage of men who are under 74 inches tall is $50\% + \frac{99.49\%}{2} \approx 99.75\%$.

(iv) If $z_0 > 0$, then then area under the normal curve *to the right* of z_0 is equal to 50%− half the table entry for z_0, because



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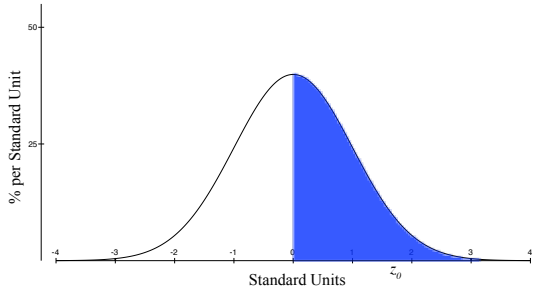

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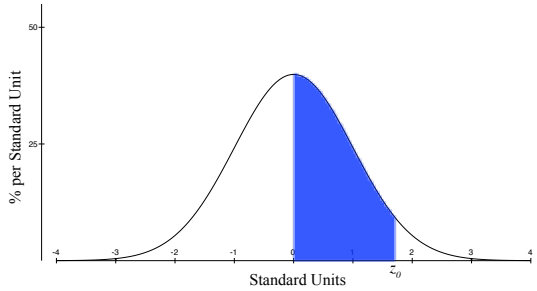


$= 50\% - \frac{1}{2}\text{Table}(z_0).$

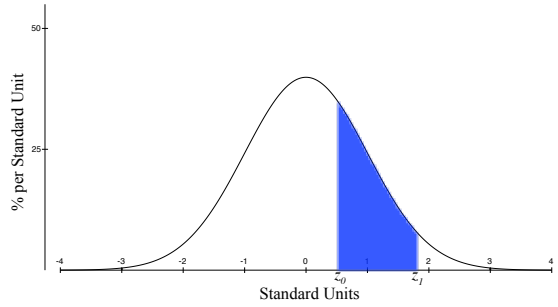
Example. What percentage of the men are taller than 68 inches?

68 inches corresponds to $\frac{68-67}{2.5} = 0.4$, so the percentage of men who are more than 68 inches tall is (approximately) $50\% - \frac{31.08\%}{2} = 34.46\%$.

(*) The areas of other types of regions under the normal curve can be calculated from the table by using (i) – (iv) and the symmetry of the normal curve around 0. For example, if $0 < z_0 < z_1$, then the area under the normal curve between z_0 and z_1 is

$$= \frac{1}{2}\text{Table}(z_1) - \frac{1}{2}\text{Table}(z_0)$$

because



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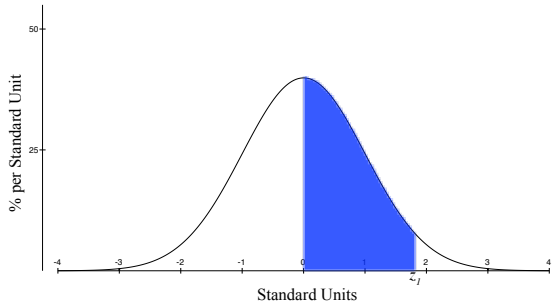

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